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Q-functions and $O_n \rightarrow S_n$ branching rules for ordinary and spin irreps

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Abstract. Following Morris and Luan Dehuai and Wybourne a simple method is given for the embedding $O_n \rightarrow S_n$ of ordinary and spin irreps in both n-dependent notation and in an n-independent reduced notation. Basic spin irrep and ordinary irreps are combined using the properties of Q-functions and raising operators in order to give a complete set of branching rules of $O_n \rightarrow S_n$ for spin irreps. The modification rules for Q-functions given by Morris are redefined to yield a complete and unambiguous set of rules.

1. Introduction

The orthogonal groups O_n and the symmetric groups S_n play an important role in nuclear and molecular models.

Corresponding to every symmetric group S_n, there exist two spin groups Γ_n and Γ'_n. The characters of Γ'_n are simply related to Γ_n (Morris 1962a, b, 1977). It has also been shown that the $2(n!)$-order group Γ_n is isomorphic to the linear fractional substitution representation group (Schur 1911) of S_n.

The calculation of the characters of Γ_n has been much studied. Although the decomposition $O_n \rightarrow S_n$ for ordinary irreps has been known for a long time the corresponding problem for spin irreps has received scant attention. Recently (Luan Dehuai and Wybourne 1981) some techniques have been developed for the resolution of Kronecker products involving the spin representations of S_n using Q-functions and Young's raising operators.

In this paper we shall first review the labelling of ordinary and spin irreps of O_n and S_n in n-dependent and in reduced notation. We then present some simple techniques of $O_n \rightarrow S_n$ branching for ordinary irreps. The $O_n \rightarrow S_n$ branching for spin irreps is simplified and obtained in the form of a product of the basic spin irrep and ordinary irreps of S_n. Morris (1962a, b, 1977) has given a set of rules for modification of non-standard Q-functions. Computer implementation of these rules showed the need for a careful specification of these rules that would cater for all possibilities. We give such a specification of the modification rules. Finally we combine basic spin irreps and ordinary irreps to complete the $O_n \rightarrow S_n$ branching rule for spin irreps. Some examples are given.

2. Reduced notation and labelling of representations of O_n and S_n

The concept of reduced notation for the labelling of representations was introduced by Murnaghan (1937) and was later used by Littlewood (1950) for the calculation of

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inner plethysms and Kronecker products by treatment of the symmetric group S_n as a subgroup of the linear group L_n.

The tensor irreps of O_n are usually labelled by $[\lambda]$ and the spin irreps by $[\Delta; \lambda]$ where $\Delta \equiv [\Delta; 0]$ is the basic spin irrep.

The tensor irreps of S_n are labelled by $\{\lambda\}$ and can be written as $[n-m, \mu]$ where (μ) is a partition of m. In reduced notation $\{\lambda\}$ may then be written as $\langle\mu\rangle$. For example an irrep $\langle 21\rangle$ in reduced notation corresponds to $\{321\}$ for S_6.

For the sake of simplicity, we shall use the same notation for the spin irreps of S_n as used for O_n. For example a spin irrep in reduced notation will be labelled by $\langle\Delta; 21\rangle$ where Δ is the basic spin irrep of S_n. No confusion arises here in the use of Δ because under $O_n \to S_n \Delta \to \Delta$. It is important to note here that $\langle\Delta; 21\rangle$ corresponds to $\langle 321\rangle'$ for S_6 in Morris' notation (Morris 1962a, b).

If $(n-k-1)$ is even then the spin irrep of S_n is self-associated, otherwise it will split into an associated pair of spin irreps designated as $\{\Delta; \lambda\}_+$ and $\{\Delta; \lambda\}_-$, where k is the length of the partition (λ) in the spin irrep $\langle\Delta; \lambda\rangle$. For example, the spin irrep $\langle\Delta; 21\rangle$ will form an associated pair $\{\Delta; 21\}_+$ and $\{\Delta; 21\}_-$ for S_6. Similarly for n even the basic spin irrep Δ will split into an associated pair Δ_+ and Δ_-.

The dimension formula for the spin irreps of S_n has been given by Schur (1911). in terms of the reduced notation $\langle\Delta; \mu\rangle$ we have

$$f_n^{\langle\Delta;\mu\rangle} = 2^{[(n-r-1)/2]} \prod_{x=0}^{m-1} (n-x) \prod_{i=1}^{r} \frac{1}{\mu_i!} \left(\frac{n-m-\mu_i}{n-m+\mu_i}\right) \prod_{0<i<s\leq r} \left(\frac{\mu_i-\mu_s}{\mu_i+\mu_s}\right) \tag{1}$$

where m is the weight and r the number of parts of the partition (μ). Explicitly we have, for example,

$$f_n^{\langle\Delta;321\rangle} = \frac{2^{[(n-4)/2]}}{360} n(n-1)(n-2)(n-7)(n-8)(n-9) \tag{2}$$

which holds for all n with the understanding that for associated pairs it gives the dimension of a single member of the pair.

3. $O_n \to S_n$ branching rule for tensor irreps

S_n is treated as a subgroup of O_n. Consider the embedding

$$[1] \downarrow \langle 1\rangle + \langle 0\rangle.$$

One can decompose an ordinary irrep $[\lambda]$ of O_n into irreps of S_n using the inner plethysm

$$(\langle 1\rangle + \langle 0\rangle) \otimes [\lambda].$$

Techniques developed by King (1975) readily lead to the result

$$[\lambda] \downarrow \langle 1\rangle \otimes \{\lambda/G\} \tag{3}$$

where

$$G = \sum_{\varepsilon} (-1)^{(e-r)/2} \{\varepsilon\}$$

where ε is a self-conjugate partition of weight e with r non-zero parts.

Plethysms of the type $\langle 1\rangle \otimes \{\mu\}$ may be evaluated by noting that any S-function $\{\mu\}$ may be expanded as the product of S-functions of the type $\{1^x\}$ such that $\{1^x\} = a_x$ where a_x is an elementary symmetric function $\Sigma \alpha_1 \alpha_2 \ldots \alpha_x$ and

$$\{\mu\} = |a_{\tilde{\mu}_s - s + t}|$$

where $\tilde{\mu}$ is the partition conjugate to μ and by using the identity

$$\langle 1\rangle \otimes \{1^r\} = \langle 1^r\rangle.$$

For example an irrep [321] of O_n is branched to S_n in reduced notation as

$$\begin{aligned}[321] \to \langle 41\rangle + 2\langle 4\rangle + \langle 321\rangle + 3\langle 32\rangle + 3\langle 311\rangle + 9\langle 31\rangle + 6\langle 3\rangle + 3\langle 221\rangle \\ + 7\langle 22\rangle + \langle 2111\rangle + 9\langle 211\rangle + 15\langle 21\rangle + 6\langle 2\rangle + 2\langle 1111\rangle + 6\langle 111\rangle + 6\langle 11\rangle + 2\langle 1\rangle.\end{aligned}$$

The above calculation is laborious but was readily evaluated using the program SCHUR which rapidly evaluates reduced products.

For a particular value of n we may convert from the reduced notation to S-functions of weight n. If $n \geqslant 2|\lambda|$ then the resulting S-functions will be in standard form. In $n < 2|\lambda|$ then the resulting S-functions are modified using the following modification rules (Littlewood 1950).

(1) In any S-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part increased by unity, the S-function being thereby changed in sign, i.e.

$$\{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_k\} = -\{\lambda_1, \ldots, \lambda_{i+1} - 1, \lambda_i + 1, \ldots, \lambda_k\}.$$

(2) In any S-function if any part exceeds by unity the preceding part, the value of the S-function is zero, i.e.

$$\text{if} \qquad \lambda_{i+1} = \lambda_i + 1 \qquad \text{then} \qquad \{\lambda_1, \ldots, \lambda_i, \lambda_{i+1}, \ldots, \lambda_k\} = 0.$$

(3) The value of any S-function is zero if the last part is negative. As examples of the application of the above rules we have:

$$\{312\} \to \text{zero} \qquad \{314\} \to -\{332\} \qquad \{3-14\} \to -\{33\} \qquad \{35-11\} \to \{44\}.$$

4. $O_n \to S_n$ branching rule for spin irreps

A spin irrep of O_n can be written as the product of the basic spin irrep and ordinary irreps, i.e.

$$[\Delta; \lambda] = \Delta \cdot [\lambda/P] \tag{4}$$

where P is a S-function series (King 1975)

$$P = \sum_m (-1)^m \{m\}$$

where (m) is a partition of one part only.

The ordinary irreps of O_n can be decomposed into ordinary irreps of S_n using equation (3). Hence

$$[\Delta; \lambda] \downarrow \Delta \cdot \langle 1\rangle \otimes \{\lambda/PG\} \equiv \Delta \cdot \langle 1\rangle \otimes \{\lambda/A\} \tag{5}$$

where A is an S-function series (King 1975)

$$A = \sum_\alpha (-1)^{a/2} \{\alpha\}.$$

where a is the weight of the partition and (α) are the partitions in Frobenius notation such that

$$(\alpha)=\begin{pmatrix} a_1 & a_2 & \dots & a_r \\ a_1+1 & a_2+1 & \dots & a_r+1 \end{pmatrix}.$$

For example $[\Delta; 321]$ is decomposed into S_n as

$$[\Delta;321]\to\Delta\cdot(\langle 41\rangle+\langle 4\rangle+\langle 321\rangle+2\langle 32\rangle+2\langle 311\rangle+4\langle 31\rangle+\langle 3\rangle+2\langle 221\rangle$$
$$+3\langle 22\rangle+\langle 2111\rangle+4\langle 211\rangle+4\langle 21\rangle+\langle 2\rangle+\langle 1111\rangle+\langle 111\rangle+\langle 11\rangle).$$

5. Raising operators and Q-functions

Young (1932) introduced raising operators for the calculation of the homogeneous product sum, h_λ, in terms of S-functions. Later on the raising operators were used for the evaluation of Q-functions (Littlewood 1961, Macdonald 1979). A raising operator δ_{ij} when operated on a partition (λ), increases λ_i by one and decreases λ_j by one provided $1\leq i<j\leq k$ where k is the length of the partition. Hence (Thomas 1981)

$$\{\lambda\}_h=\prod_{1\leq i<j\leq k}(1-\delta_{ij})h_\lambda \qquad \text{where } h_\lambda\equiv h_{\lambda_2}\dots h_{\lambda_k}$$

and

$$h_\lambda=\prod_{1\leq i<j\leq k}\frac{1}{(1-\delta_{ij})}\{\lambda\}_h \qquad \text{where } \{\lambda\}_h=|h_{\lambda_i-i+j}|.$$

More recently Luan Dehuai and Wybourne (1981) introduced a special raising operator δ_{0j} for the calculation of Kronecker products of the basic spin irrep with ordinary irreps of S_n in reduced notation.

The raising operators may be used to express the generalised S-functions $\{\lambda\}_q=|q_{\lambda_i-i+j}|$ in terms of Q-functions (Littlewood 1961, Thomas 1977, 1981)

$$\{\lambda\}_q=\prod_{i<j}(1+\delta_{ij})Q_\lambda \tag{6}$$

and vice versa

$$Q_\lambda=\prod_{i<j}\frac{1}{(1+\delta_{ij})}\{\lambda\}_q. \tag{7}$$

At this stage it is important to note the following points regarding the use of raising operators. First of all the raising operators $\Pi_{i<j}(1+\delta_{ij})$ or $\Pi_{i<j}(1+\delta_{ij})^{-1}$ are not commutative, so it is important to keep the order. Secondly the raising operators may generate non-standard partitions which should be standardised after completing the operation. However, partitions having a negative last part cannot generate any non-vanishing partitions and hence may be discarded as they arise. The modification rules are used to standardise the non-standard partition. The rules given by Morris (Morris 1962a, b) are incomplete. Using equations (3.3.2)-(3.3.4) of Morris (1962a) the following rules have been derived. Any list of Q-functions may be converted into a list of standard Q-functions by sequential application of the following four rules to the list.

(i) The parts of the Q-function are first ordered so that the absolute magnitude of the parts are in descending order when read from left to right. This is achieved by repeated use of

$$Q_{(\ldots\lambda_i,\lambda_{i+1},\ldots)} = -Q_{(\ldots\lambda_{i+1},\lambda_i,\ldots)}$$

whenever $|\lambda_{i+1}| > |\lambda_i|$, remembering that $Q_{(\mu,0)} = Q_{(\mu)}$.

(ii) Q-functions with consecutive repeated parts are null.

(iii) Q-functions where a negative part $-\lambda_p$ precedes λ_p is modified by use of the identity

$$Q_{(\lambda_1\ldots-\lambda_p,\lambda_p\ldots\lambda_k)} = (-1)^{\lambda_p} 2Q_{(\lambda_1\ldots\lambda_k)}.$$

(iv) Any remaining Q-function containing a negative part is null.

Thus application of (i) leads to $Q_{(304-212-11)} \to Q_{(43-221-11)}$. Application of (ii) then leads to $Q_{(43-221-11)} \to -4Q_{(431)}$ and hence $Q_{(304-212-11)} \equiv -4Q_{(431)}$.

Likewise one readily finds that

$$Q_{(1003-11)} \to 2Q_{(31)} \qquad Q_{(3211)} \to \text{zero}$$

$$Q_{(32-221-11)} \to -4Q_{(321)} \qquad Q_{(320-212-11)} \to -4Q_{(321)}.$$

We note in the above examples that it is possible to have repeated parts provided they are separated by a negative part of the same magnitude. Inspection of equation (2) gives the dimension results

$$f_4^{\langle\Delta;321\rangle} = -4 \qquad f_5^{\langle\Delta;321\rangle} = -4 \qquad f_6^{\langle\Delta;321\rangle} = -4 \qquad f_7^{\langle\Delta;321\rangle} = 0$$

which are respectively consistent with the Q-function equivalences

$$Q_{(-2321)} = -2Q_{(31)} \qquad Q_{(-1321)} = -2Q_{(32)}$$

$$Q_{(0321)} = -Q(321) \qquad \text{and} \qquad Q_{(1321)} = 0.$$

6. Completion of the $O_n \to S_n$ spin branching rule

In the previous sections we have developed the technique of branching an irreducible spin representation of O_n to S_n in the form of the product of the basic spin irrep and ordinary irreps of S_n. In order to combine the basic spin irrep and the ordinary irreps we note that the spin characters of S_n are related to the Q-functions (Schur 1911) as

$$Q_{(\lambda)} = 2^{(k+p+\varepsilon)/2} \sum_{(\pi)} \frac{h_\pi}{h} \zeta_\pi^{[\Delta;\lambda]} S_\pi \tag{8}$$

where $\zeta_\pi^{[\Delta;\lambda]}$ is a simple spin character of the class $(\pi) = (1^{\alpha_1}3^{\alpha_3}\ldots)$ involving odd cycles only, $p = \alpha_1 + \alpha_3 + \ldots$, h_π is the order of the class (π), h is the order of Γ_n, $S_\pi = S_1^{\alpha_1} S_3^{\alpha_3}\ldots$ and $\varepsilon = 0$ or 1 according as $(n-k)$ is even or odd.

Now the problem is the resolution of the inner product of Q-functions with S-functions $\{\lambda\}_h$. We note that products of this type will produce S-functions $\{\lambda\}_q$ which in turn can be changed into Q-functions using (6) (Luan Dehuai and Wybourne 1981). So we have

$$Q_{(n)} \circ \{\lambda\}_h = \prod_{i<j} (1+\delta_{ij}) Q_{(\lambda)} \tag{9}$$

where $Q_{(n)}$ represents the n-dependent basic spin irrep. In reduced notation the above equation is written as

$$Q_{\langle 0\rangle} \circ \langle \mu \rangle = \prod_{0 \leq i < j} (1+\delta_{ij}) Q_{\langle \mu \rangle} \tag{10}$$

and we have noted the theorems and lemmas of Thomas (1977) together with the fact that $Q_{(n)} = q_n$.

Equation (10) gives us a set of Q-functions. These Q-functions can be converted into the irreducible spin representations of any particular group of S_n by making use of the following.

(i) Replace every Q-function by $\langle \Delta; \mu \rangle$.

(ii) Multiply them by a factor $2^{[(k-n(\mathrm{mod}\,2))/2]}$.

If $(n-k-1)$ is odd then modify as

$$\{\Delta; \mu\} = \{\Delta; \mu\}_+ + \{\Delta; \mu\}_-. \tag{11}$$

Thus our final algorithm for evaluating the branching rule $O_n \to S_n$ for spin irreps $[\Delta; \lambda]$ of O_n is as follows.

(1) Evaluate the terms, in reduced notation, contained in

$$\langle 1 \rangle \otimes \{\lambda / A\}.$$

(2) Apply the raising operator $\Pi(1+\delta_{ij})$ to the terms produced in (1) and standardise the resulting partitions as Q-functions and then replace each Q-function $Q_{\langle \lambda \rangle}$ by $\langle \Delta; \lambda \rangle$.

(3) For a particular value of n multiply each $\langle \Delta; \lambda \rangle$ by $2^{[(k-n(\mathrm{mod}\,2))/2]}$ and replace $\langle \Delta; \lambda \rangle$ by $\{\Delta; \lambda\}$ and modify using (11).

The above algorithm has been incorporated in the program SCHUR. The branching rule is calculated first performing steps (1) and (2) to yield an n-independent list of $\langle \Delta; \lambda \rangle$. Thus for $[\Delta; 432]$ we obtain the list given below:

$$\begin{aligned}
&2\langle \Delta; 71\rangle + 10\langle \Delta; 7\rangle + \langle \Delta; 63\rangle + \langle \Delta; 621\rangle + 14\langle \Delta; 62\rangle + 47\langle \Delta; 61\rangle + 93\langle \Delta; 6\rangle \\
&\quad + \langle \Delta; 54\rangle + 2\langle \Delta; 531\rangle + 22\langle \Delta; 53\rangle + 14\langle \Delta; 521\rangle + 118\langle \Delta; 52\rangle + 234\langle \Delta; 51\rangle \\
&\quad + 312\langle \Delta; 5\rangle + \langle \Delta; 432\rangle + 13\langle \Delta; 431\rangle + 76\langle \Delta; 43\rangle + 56\langle \Delta; 421\rangle + 328\langle \Delta; 42\rangle \\
&\quad + 506\langle \Delta; 41\rangle + 544\langle \Delta; 4\rangle + 71\langle \Delta; 321\rangle + 350\langle \Delta; 32\rangle + 556\langle \Delta; 31\rangle + 560\langle \Delta; 3\rangle \\
&\quad + 282\langle \Delta; 21\rangle + 350\langle \Delta; 2\rangle + 124\langle \Delta; 1\rangle + 38\langle \Delta; 0\rangle.
\end{aligned}$$

This is a universal list in the sense that once it is calculated it holds for all n. It is step (3) that specialises the list to a particular value of n. We give below the cases of $n = 16$ and 17.

Group is O(16) $\quad \to \dim[\Delta; 432] = 4635\,158\,528.$

Group is S(16)

$$\begin{aligned}
&4\{\Delta; 71\}_+ + 4\{\Delta; 71\}_- + 20\{\Delta; 7\} + 2\{\Delta; 63\}_+ + 2\{\Delta; 63\}_- + 4\{\Delta; 621\} \\
&\quad + 28\{\Delta; 62\}_+ + 28\{\Delta; 62\}_- + 94\{\Delta; 61\}_+ + 94\{\Delta; 61\}_- \\
&\quad + 186\{\Delta; 6\} + 2\{\Delta; 54\}_+ + 2\{\Delta; 54\}_- + 8\{\Delta; 531\} + 44\{\Delta; 53\}_+ + 44\{\Delta; 53\}_- \\
&\quad + 56\{\Delta; 521\} + 236\{\Delta; 52\}_+ + 236\{\Delta; 52\}_- + 468\{\Delta; 51\}_+ + 468\{\Delta; 51\}_- \\
&\quad + 624\{\Delta; 5\} + 4\{\Delta; 432\} + 52\{\Delta; 431\} + 152\{\Delta; 43\}_+ + 152\{\Delta; 43\}_- \\
&\quad + 224\{\Delta; 421\} + 656\{\Delta; 42\}_+ + 656\{\Delta; 42\}_- + 1012\{\Delta; 41\}_+ + 1012\{\Delta; 41\}_-
\end{aligned}$$

$$+1088\{\Delta;4\}+284\{\Delta;321\}+700\{\Delta;32\}_{+}+700\{\Delta;32\}_{-}+1112\{\Delta;31\}_{+}$$
$$+1112\{\Delta;31\}_{-}+1120\{\Delta;3\}+564\{\Delta;21\}_{+}+564\{\Delta;21\}_{-}+700\{\Delta;2\}$$
$$+248\{\Delta;1\}+38\{\Delta;0\}_{+}+38\{\Delta;0\}_{-}.$$

Dimension = 4635 158 528.

Group is O(17) $\rightarrow \dim[\Delta;432]=8364\,195\,840.$

Group is S(17)

$$4\{\Delta;71\}+10\{\Delta;7\}_{+}+10\{\Delta;7\}_{-}+2\{\Delta;63\}+2\{\Delta;621\}_{+}+2\{\Delta;621\}_{-}+28\{\Delta;62\}$$
$$+94\{\Delta;61\}+93\{\Delta;6\}_{+}+93\{\Delta;6\}_{-}+2\{\Delta;54\}+4\{\Delta;531\}_{+}+4\{\Delta;531\}_{-}$$
$$+44\{\Delta;53\}+28\{\Delta;521\}_{+}+28\{\Delta;521\}_{-}+236\{\Delta;52\}+468\{\Delta;51\}$$
$$+312\{\Delta;5\}_{+}+312\{\Delta;5\}_{-}+2\{\Delta;432\}_{+}+2\{\Delta;432\}_{-}+26\{\Delta;431\}_{+}$$
$$+26\{\Delta;43\}_{-}+152\{\Delta;43\}_{+}+112\{\Delta;421\}_{+}+112\{\Delta;421\}_{-}+656\{\Delta;42\}$$
$$+1012\{\Delta;41\}+544\{\Delta;4\}_{+}+544\{\Delta;4\}_{-}+142\{\Delta;321\}_{+}+142\{\Delta;321\}_{-}$$
$$+700\{\Delta;32\}+1112\{\Delta;31\}+560\{\Delta;3\}_{+}+560\{\Delta;3\}_{-}+564\{\Delta;21\}$$
$$+350\{\Delta;2\}_{+}+350\{\Delta;2\}_{-}+124\{\Delta;1\}_{+}+124\{\Delta;1\}_{-}+38\{\Delta;0\}.$$

Dimension = 8364 195 840.

These rather large examples show the power of the methods outlined which avoid all use of character tables.

It is worth noting that Stembridge (1989) has given an alternative presentation for the inner product of the basic spin irrep with an arbitrary tensor irrep of S_n that avoids the need for modification rules.

8. Concluding remarks

We have completed the statement of the modification rules for Q-functions and stated systematic algorithms for evaluating $O_n \rightarrow S_n$ branching rules for both ordinary and spin irreps of O_n.

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